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Extensional Vibrations of Piezoelectric Plates

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SUMMARY

We consider piezoelectric plates with a thickness small with respect to the lateral dimensions. The surfaces of these plates are partly coated with electrodes. Equations are derived which describe the lateral vibrations of these plates approximately. The first resonance-frequency of a circular plate as a function of the radius of the electrodes is computed and compared with measured values.

1. Introduction

In this paper extensional vibrations of piezoelectric plates are investigated. Piezoelectricity is a reversible, electromechanical phenomenon. In materials exhibiting this effect, stresses and strains occur when an electric field is applied and inversely, mechanical stresses produce electric polarization and hence an electric field [1], [2].

The faces of the plates considered here are partly coated with electrodes, situated symmetrically with respect to the middle plane (fig. 1). Two parts of the plate can be distinguished, a part with faces which are free of electrodes (I) and a part with electroded faces (II). When an alternating potential difference is applied to the electrodes, extensional vibrations occur. The middle plane in between the faces remains flat. In this paper a simple theory is stated for low-frequency modes. The linear, three-dimensional equations for a piezoelectric body are reduced to two-dimensional ones. The two-dimensional approach is based on exact solutions for infinite plates both free of electrodes and completely covered by electrodes. By applying suitable boundary conditions we "match" the solutions for both parts of the plate.

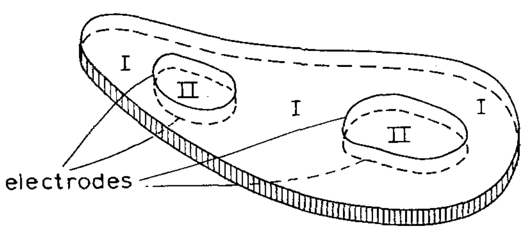

Figure 1. Piezoelectric plate with electrodes.

This theory is applied to a circular piezoceramic plate covered by concentric circular electrodes. The first resonance frequency is computed as a function of the ratio of the radius of the electrodes to the radius of the plate. These numerical results are checked by experiments.

2. The linear Equations of Piezoelectricity

Throughout this paper the stresses, strains, electric field and electric displacements are supposed to be small in order that the linear piezoelectric equations can be applied. These are given in tensor notation. A vertical line followed by an index denotes covariant differentiation, a comma followed by an index ordinary differentiation and a dot differentiation with respect to time. The summation convention for repeated indices is employed. Latin indices range over

1, 2 and 3; Greek indices over 1 and 2. In general coordinates θ^1, θ^2, θ^3 the basic piezoelectric equations are:

$$T^{ij} = {}^E c^{ijkl} S_{kl} - e^{kij} E_k \,, \tag{1}$$

$$D^i = e^{ikl} S_{kl} + {}^S\varepsilon^{ik} E_k \,, \tag{2}$$

in which the S_{kl} represent the strain-tensor, the D_k the electric displacements, the T_{kl} the stress-tensor and the E_k the electric field. The elastic coefficients are denoted by ${}^E c^{ijkl}$, the piezoelectric ones by e^{ikl} and the dielectric ones by ${}^S\varepsilon^{ik}$. Equations (1) and (2) are algebraically equivalent with:

$$S_{ij} = {}^E S_{ijkl} T^{kl} + d^{k}_{.ij} E_k \,, \tag{1a}$$

$$D^i = d^{i}_{.kl} T^{kl} + {}^T\varepsilon^{ik} E_k \,. \tag{2a}$$

In addition to these equations the following ones are valid. The equations of motion:

$$T^{ij}|_j = \rho \ddot{U}^i \tag{3}$$

where ρ represents the density and U^i the displacement in the θ^i direction. The strain-displacement relations:

$$S_{ij} = \tfrac{1}{2}(U_i|_j + U_j|_i) \,. \tag{4}$$

Finally, with restriction to the quasi-static treatment of the electric field, the Maxwell equations of electrostatics:

$$D^i|_i = 0 \,, \tag{5}$$

$$E_i = -V_{,i} \,. \tag{6}$$

The equation (1)–(6) hold inside the piezoelectric material. Although polarization plays a fundamental role in piezoelectricity, it does not appear explicitly here. We restrict ourselves to the remark that the polarization can be expressed linearly in the electric field and the electric displacement.

At the boundary of the body the following transition-conditions must be satisfied. Denoting the tangential component of the electric field and the outward directed normal component of the electric displacement inside and outside the body by $E^{(i)}_{(t)}$, $E^{(o)}_{(t)}$ and $D^{(i)}_{(n)}$, $D^{(o)}_{(n)}$, we have:

$$E^{(i)}_{(t)} = E^{(o)}_{(t)} \,, \tag{7}$$

$$D^{(o)}_{(n)} - D^{(i)}_{(n)} = F \,, \tag{8}$$

where F represents the free charge on the boundary.

3. Exact Solution for the Infinite Plate

We consider an infinitely extended piezoelectric plate in vacuum described by cartesian coordinates x_1, x_2, x_3. The faces of the plate are given by $x_3 = \pm h$ (h constant) and they are either free of electrodes or completely covered by electrodes. The faces $x_3 = \pm h$ are traction-free.

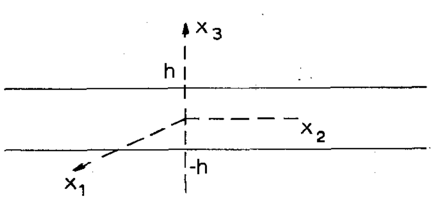

Figure 2. The infinite plate.

In case electrodes are present a constant charge distribution on the electrodes will be prescribed, q on the upper one and $-q$ on the lower one. The plate is loaded at infinity by constant stresses T^{11}, T^{12} and T^{22}. In this section a solution with constant stresses, strains, electric field and electric displacements will be found which satisfies the transition—and boundary—conditions.

Now the T^{j3} become zero throughout the plate, since they vanish at $x_3 = \pm h$. The $T^{\alpha\beta}$ are equal to the prescribed stresses at infinity.

The electric field outside the plate is the sum of a field due to the charge on the electrodes and a field due to the polarization inside the plate. Since the polarization is constant and the charge densities are constant and opposite, the field outside the plate vanishes and the electric displacements outside the plate vanish too [4]. Transition-condition (7) and (8) now yield:

$$E_\alpha = 0 ; \; D^3 = -q , \tag{9a), (9b}$$

inside the plate. When no electrodes are present we have $D^3 = q = 0$.

For $i=3$ equation (2a) reads:

$$-q = d^3_{.\alpha\beta} T^{\alpha\beta} + {}^T\varepsilon^{33} E_3 , \tag{10}$$

which we use to express E_3 in q and $T^{\alpha\beta}$. Substituting this expression in (1a) and the remaining equations of (2a) we find:

$$S_{ij} = \left({}^E S_{ij\alpha\beta} - \frac{d^3_{.ij} d^3_{.\alpha\beta}}{{}^T\varepsilon^{33}} \right) T^{\alpha\beta} - \frac{d^3_{.ij}}{{}^T\varepsilon^{33}} q ; \tag{11}$$

$$D^\lambda = \left(d^\lambda_{.\alpha\beta} - \frac{{}^T\varepsilon^{\lambda 3} d^3_{.\alpha\beta}}{{}^T\varepsilon^{33}} \right) T^{\alpha\beta} - \frac{{}^T\varepsilon^{\lambda 3}}{{}^T\varepsilon^{33}} q . \tag{12}$$

Now the electric field components, the strains and electric displacements are expressed in the known quantities $T^{\alpha\beta}$ and q. Equation (6) gives:

$$V = -E_3 x_3 . \tag{13}$$

Equation (4) can be solved for the displacements. Assuming at the origin:

$$u_1 = u_2 = u_3 = 0 , \tag{14a}$$

$$u_{3,1} = u_{3,2} = u_{2,1} = 0 , \tag{14b}$$

in order to avoid a rigid body motion, they become:

$$\begin{aligned} u_1 &= S_{11} x_1 + 2S_{12} x_2 + 2S_{13} x_3 \\ u_2 &= \qquad\quad S_{22} x_2 + 2S_{23} x_3 \\ u_3 &= \qquad\qquad\qquad S_{33} x_3 . \end{aligned} \tag{15}$$

We conclude that the middle plane remains flat. The solution given above satisfies the equations and boundary-conditions mentioned in section 2.

4. Equations for a Thin, Partly Electroded Piezoelectric Plate

A piezoelectric plate partly coated with electrodes on the surfaces is considered (fig. 1). The lateral dimensions of the parts I and II will be large with respect to the thickness. A two-dimensional theory for the extension of this plate is derived in this section. We first consider part II.

The position vector $\boldsymbol{R}$ of a point of the plate will have the form [5], page 185:

$$\boldsymbol{R} = \boldsymbol{r}(\theta^1, \theta^2) + \theta^3 \boldsymbol{a}_3 \tag{16}$$

where $\theta^3 = 0$ represents the middle plane of the plate; $\boldsymbol{a}_3$ is a constant unit vector perpendicular to the middle plane. We assume that the local distribution of stresses, electric field and electric displacements over the thickness are the same as for the infinite plate. Hence these quantities will be independent of θ^3, and the T^{j3} and E_α are neglected, while $E_3 = -(2h)^{-1} V$ and $D^3 = -q$. Here V represents the potential difference between the faces $\theta^3 = \pm h$. Equations (1) yield for $j=3$:

$$0 = {}^E c^{i3kl} S_{kl} + (2h)^{-1} e^{3i3} V . \tag{17}$$

By equations (17) the S_{j3} are expressed in $S_{\alpha\beta}$ and V. Hence they will not appear explicitly in the plate equations. In virtue of the elimination of the S_{j3}, the equations (1) and (2) transform into:

$$T^{\alpha\beta} = c_{(1)}^{\alpha\beta\gamma\delta} S_{\gamma\delta} + (2h)^{-1} e_{(1)}^{3\alpha\beta} V \,, \tag{18}$$

$$-q = e_{(1)}^{3\gamma\delta} S_{\gamma\delta} - (2h)^{-1} \varepsilon_{(1)}^{33} V \,, \tag{19}$$

where $c_{(1)}^{\alpha\beta\gamma\delta}$, $e_{(1)}^{3\alpha\beta}$ and $\varepsilon_{(1)}^{33}$ represent the elastic, piezoelectric and dielectric constants after elimination of the S_{j3}.

Instead of the equations of motion for an infinitesimal three-dimensional element, we will consider the equations of motion for the plate-element given in fig. 3 (Since we consider

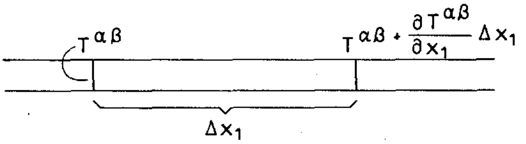

Figure 3. Element of the plate.

extensional motions only, the equations of motion in the $\boldsymbol{a}_3$ direction is neglected). The equations are obtained by integrating equations (3) for $i=1, 2$ over the thickness:

$$\frac{1}{2h}\int_{-h}^{h} T^{\alpha\beta}|_\beta d\theta^3 + \{T^{\alpha 3}_{(\theta^3=h)} - T^{\alpha 3}_{(\theta^3=-h)}\} = \frac{\rho}{2h}\int_{-h}^{h} \ddot{U}^\alpha d\theta^3 \,. \tag{20}$$

The $T^{\alpha 3}_{(\theta^3=\pm h)}$ vanish and the integrand of the integral at the left-hand side is supposed to be independent of θ^3. Introducing the mean of U^α, denoted by $\bar{U}^\alpha$, the right-hand side reads $\rho\ddot{\bar{U}}^\alpha$, hence:

$$T^{\alpha\beta}|_\beta = \rho\ddot{\bar{U}}^\alpha \,. \tag{21}$$

Since the $S_{\alpha\beta}$ are assumed to be constant over the thickness, integration of equation (4) yields:

$$S_{\alpha\beta} = \tfrac{1}{2}(\bar{U}_\alpha|_\beta + \bar{U}_\beta|_\alpha) \,. \tag{22}$$

Since V is constant in part II, substitution of (22) and (18) into (21) now gives:

$$c_{(1)}^{\alpha\beta\gamma\delta} \bar{U}_\gamma|_{\delta\beta} = \rho\ddot{\bar{U}}^\alpha \,. \tag{23}$$

The relation between the total charge Q on the upper electrode and the potential V follows from (19)

$$\begin{aligned} Q &= \iint q\, dA = \iint \{-e_{(1)}^{3\gamma\delta} S_{\gamma\delta} + (2h)^{-1} \varepsilon_{(1)}^{33} V\}\, dA \\ &= -\iint e_{(1)}^{3\gamma\delta} \bar{U}_\gamma|_\delta dA + (2h)^{-1} \varepsilon_{(1)}^{33} VA \\ &= -\int e_{(1)}^{3\gamma\delta} \bar{U}_\gamma \nu_\delta ds + (2h)^{-1} \varepsilon_{(1)}^{33} VA \end{aligned} \tag{24}$$

in which A represents the area of the electrode, $\iint dA$ denotes integration over this area and $\int ds$ integration over its boundary. The ν_δ represent the covariant components of the outward directed unit normal vector on the boundary [5], page 31.

In part I the equations (18), (19), (21) and (22) holds also, if $q=0$. Elimination of V from (18) and (19) yields:

$$T^{\alpha\beta} = c_{(2)}^{\alpha\beta\gamma\delta} S_{\gamma\delta} \,, \tag{25}$$

where

$$c_{(2)}^{\alpha\beta\gamma\delta} = c_{(1)}^{\alpha\beta\gamma\delta} + \frac{e_{(1)}^{3\alpha\beta} e_{(1)}^{3\gamma\delta}}{\varepsilon_{(1)}^{33}} \,. \tag{25a}$$

Substitution of (22) and (25) into (21) now yields:

$$c_{(2)}^{\alpha\beta\gamma\delta} U_\gamma|_{\delta\beta} = \rho \ddot{U}^\alpha . \tag{26}$$

The equations (23) and (26) are matched on the boundary of part I and part II on condition that the stresses and the displacements must be continuous at this boundary. By means of an energy consideration as in [6] uniqueness of solution can be proved provided the following conditions are satisfied:

(1) on the boundary of the plate either the stresses or the displacements are prescribed.

(2) on the electrodes either the potential V or the total charge Q is prescribed.

It can be seen that the piezoelectric effect is only noticeable in the coefficients of the equations (24) and (26) but not in the coefficients of (23). Hence, in case of a fully electroded plate the piezoelectric effect appears only in the stress-boundary conditions according to equations (18).

5. Rotational Symmetric Vibrations of a Circular Ceramic Plate

As an application of the equations derived in Section 4 we consider a circular ceramic plate with radius R. In order to achieve rotational symmetry, the direction of polarization of the ceramic is perpendicular to the plane of the plate. Only the radial, rotational symmetric vibrations of this plate are considered. We first consider a plate with completely electroded surfaces.

In cartesian coordinates the coefficients of ceramic material can be denoted by the following tables. The elastic coefficients:

$$\begin{matrix} {}^E S_{1111} & {}^E S_{1122} & {}^E S_{1133} & 0 & 0 & 0 \\ & {}^E S_{1111} & {}^E S_{1133} & 0 & 0 & 0 \\ & & {}^E S_{3333} & 0 & 0 & 0 \\ & & & {}^E S_{1313} & 0 & 0 \\ & & & & {}^E S_{1313} & 0 \\ & & & & & \frac{1}{2}({}^E S_{1111} - {}^E S_{1122}) \end{matrix}$$

The piezoelectric coefficients:

$$\begin{matrix} 0 & 0 & 0 & 0 & d^1_{.13} & 0 \\ 0 & 0 & 0 & d^1_{.13} & 0 & 0 \\ d^3_{.11} & d^3_{.11} & d^3_{.33} & 0 & 0 & 0 \end{matrix}$$

The dielectric coefficients:

$$\begin{matrix} {}^T\varepsilon^{11} & 0 & 0 \\ 0 & {}^T\varepsilon^{11} & 0 \\ 0 & 0 & {}^T\varepsilon^{33} \end{matrix}$$

Since T^{j3} and E_α are neglected equations (1a) and (2a) reduce to:

$$\left.\begin{aligned} S_{11} &= {}^E S_{1111} T^{11} + {}^E S_{1122} T^{22} + d^3_{.11} E_3 , \\ S_{22} &= {}^E S_{1122} T^{11} + {}^E S_{1111} T^{22} + d^3_{.11} E_3 , \end{aligned}\right\} \quad S_{12} = \tfrac{1}{2}({}^E S_{1111} - {}^E S_{1122}) T^{12} , \tag{27}$$

$$D^3 = d^3_{.11}(T^{11} + T^{22}) + {}^T\varepsilon^{33} E_3 . \tag{28}$$

The elastic coefficients are usually expressed in Young's modulus $Y = ({}^E S_{1111})^{-1}$ and Poisson's ratio $\nu = -Y {}^E S_{1122}$. An inversion of these equations yields:

$$\begin{aligned} T^{11} &= \alpha_1 S_{11} + \alpha_2 S_{22} + \alpha_3 (2h)^{-1} V , \\ T^{22} &= \alpha_2 S_{11} + \alpha_1 S_{22} + \alpha_3 (2h)^{-1} V , \\ T^{12} &= \tfrac{1}{2}(\alpha_1 - \alpha_2) S_{12} , \end{aligned} \tag{29}$$

$$q \quad = -\alpha_3(S_{11}+S_{22})+\alpha_4(2h)^{-1}V\,, \tag{30}$$

where

$$\alpha_1 = \frac{Y}{1-\nu^2}\,;\quad \alpha_2 = \frac{\nu Y}{1-\nu^2}\,;$$

$$\alpha_3 = \frac{Yd^3_{.11}}{1-\nu}\,;\quad \alpha_4 = {}^{T}\varepsilon^{33}\left(1-\frac{2k_1}{1-\nu}\right).$$

The constant k_1 is the square of an electromechanical coupling-factor:

$$k_1 = \frac{Y(d^3_{.11})^2}{{}^{T}\varepsilon^{33}}\,. \tag{31}$$

We try to find a rotational symmetric solution in polar coordinates (r, φ). Denoting the radial displącement by U_r, we have:

$$U^r = U_r = U_r(r) \tag{32}$$

$$S_{rr} = U_{r,r}\,;\quad S_{\varphi\varphi} = rU_r\,;\quad S_{r\varphi} = 0\,. \tag{33}$$

After a tensor-transformation equations (29) and (30) read:

$$\left.\begin{aligned} T^{rr} &= \alpha_1 S_{rr} + \frac{\alpha_2}{r^2} S_{\varphi\varphi} + \alpha_3(2h)^{-1}V\,,\\ T^{\varphi\varphi} &= \frac{\alpha_2}{r^2} S_{rr} + \frac{\alpha_1}{r^4} S_{\varphi\varphi} + \frac{\alpha_3}{r^2}(2h)^{-1}V\,,\\ T^{r\varphi} &= \frac{\alpha_1-\alpha_2}{2r^2} S_{r\varphi}\,, \end{aligned}\right\} \tag{34}$$

$$q \quad = -\alpha_3 S_{rr} - \frac{\alpha_3}{r^2} S_{\varphi\varphi} + \alpha_4(2h)^{-1}V\,. \tag{35}$$

Substitution of (33) yields:

$$\left.\begin{aligned} T^{rr} &= \alpha_1 U_{r,r} + \frac{\alpha_2}{r} U_r + \alpha_3(2h)^{-1}V\,,\\ T^{\varphi\varphi} &= \frac{\alpha_2}{r^2} U_{r,r} + \frac{\alpha_1}{r^3} U_r + \frac{\alpha_3}{r^2}(2h)^{-1}V\,, \end{aligned}\right\} \tag{36}$$

$$q \quad = -\alpha_3\left(U_{r,r} + \frac{1}{r}U_r\right) + \alpha_4(2h)^{-1}V\,. \tag{37}$$

The equation of motion in the radial direction reads:

$$T^{rr}{}_{,r} - rT^{\varphi\varphi} + \frac{1}{r}T^{rr} = \rho\ddot{U}^r\,. \tag{38}$$

After a substitution of (36) and on the assumption of a periodic solution $U_r(r)\cos\omega\tau$ we find:

$$U_{r,rr} + \frac{1}{r}U_{r,r} + \left(\frac{\rho\omega^2}{\alpha_1} - \frac{1}{r^2}\right)U_r = 0\,. \tag{39}$$

With a change of variable $t = r(\rho/\alpha_1)^{\frac{1}{2}}\omega$:

$$t^2 U_{r,tt} + tU_{r,t} + (t^2-1)U_r = 0\,. \tag{40}$$

We use the Bessel functions J_1, Y_1 as a fundamental system of solutions. Since $U_r(0)=0$:

$$U_r(t) = C_1 J_1(t)\,. \tag{41}$$

Considering the boundary condition $T^{rr}_{(R)}=0$ we find from (36) (with $J_1'(z) = J_0(z) - z^{-1}J_1(z)$):

$$\frac{C_1}{R}\left\{\frac{\alpha_2-\alpha_1}{\alpha_1}J_1(t_0)+t_0J_0(t_0)\right\}+\frac{\alpha_3}{\alpha_1}(2h)^{-1}V=0\,, \tag{42}$$

where $t_0=R(\rho/\alpha_1)^{\frac{1}{2}}\omega$. The total charge Q on the upper electrode follows from (37):

$$\begin{aligned} Q &= \iint q\,dA=\alpha_4(2h)^{-1}V\pi R^2-\alpha_3\int_0^R\left(U_{r,r}+\frac{1}{r}U_r\right)2\pi r\,dr \\ &=\pi R\{\alpha_4(2h)^{-1}VR-2\alpha_3C_1J_1(t_0)\}\,. \end{aligned} \tag{43}$$

The equations (42) and (43) give the solution when ω and either Q or V is given.

We define the fundamental frequencies of the plate as the frequencies at which the plate is able to vibrate without external energy-supply. There are two possibilities:

1) The resonance frequencies ω_r. In this case $V=0$ and ω_r follows from (42) since $C_1\neq 0$:

$$\frac{\alpha_2-\alpha_1}{\alpha_1}J_1(t_0)+t_0J_0(t_0)=0\,. \tag{44}$$

2) The antiresonance frequencies ω_a. In this case $I=dQ/dt=0$ and hence $Q=0$ and after elimination of V from (42) and (43) we find ω_a from:

$$\frac{(\alpha_2-\alpha_1)\alpha_4+2\alpha_3^2}{\alpha_1\alpha_4}J_1(t_0)+t_0J_0(t_0)=0\,. \tag{45}$$

When we consider a plate without electrodes, q vanishes and elimination of V from (36) and (37) yields:

$$\begin{aligned} T^{rr} &= \beta_1U_{r,r}+\frac{\beta_2}{r}U_r\,, \\ T^{\varphi\varphi} &= \frac{\beta_2}{r^2}U_{r,r}+\frac{\beta_1}{r^3}U_r\,, \end{aligned} \tag{46}$$

where

$$\beta_1=\alpha_1+\frac{\alpha_3^2}{\alpha_4}=\alpha_1\frac{(1-k_1)(1-\nu)}{1-\nu-2k_1}$$

$$\beta_2=\alpha_2+\frac{\alpha_3^2}{\alpha_4}=\alpha_2\frac{(\nu+k_1)(1-\nu)}{\nu(1-\nu-2k_1)}\,.$$

The equation of motion reads:

$$s^2U_{r,ss}+sU_{r,s}+(s^2-1)U_r=0\,, \tag{47}$$

with $s=r(\rho/\beta_1)^{\frac{1}{2}}\omega$. The boundary conditions $U_r(0)=0$ and $T^{rr}(R)=0$ yield:

$$U_r=C_2J_1(s) \tag{48}$$

$$\frac{C_2\beta_1}{R}\left(\frac{\beta_2-\beta_1}{\beta_1}J_1(s_0)+s_0J_0(s_0)\right)=0 \tag{49}$$

where $s_0=R(\rho/\beta_1)^{\frac{1}{2}}\omega$. Here the external energy-supply vanishes since $Q=0$ and the fundamental frequencies follow from equating the last factor in the left-hand side of (49) to zero.

Finally we consider a plate with concentric electrodes of radius λR, $0<\lambda<1$. The solution in both parts is given by

$$\begin{aligned} U_r &= C_1J_1(t) && 0<r<\lambda R \\ U_r &= C_2J_1(s)+C_3Y_1(s) && \lambda R<r<R \end{aligned} \tag{50}$$

with the following boundary conditions: at λR the displacements must be continuous:

$$C_1J_1(t_1)-C_2J_1(s_1)-C_3Y_1(s_1)=0\,, \tag{51}$$

at λR the stress T^{rr} must be continuous:

$$
\begin{aligned}
&C_1\{(\alpha_2-\alpha_1)J_1(t_1)+\alpha_1 t_1 J_0(t_1)\}\\
&-C_2\{(\beta_2-\beta_1)J_1(s_1)+\beta_1 s_1 J_0(s_1)\}\\
&-C_3\{(\beta_2-\beta_1)Y_1(s_1)+\beta_1 s_2 Y_0(s_1)\} = -\alpha_3(2h)^{-1}V\lambda R\,,
\end{aligned}
\tag{52}
$$

at R the stress T^{rr} must vanish:

$$
\begin{aligned}
&C_2\{(\beta_2-\beta_1)J_1(s_2)+\beta_1 s_2 J_0(s_2)\}\\
&+C_3\{(\beta_2-\beta_1)Y_1(s_2)+\beta_1 s_2 Y_0(s_2)\}=0\,,
\end{aligned}
\tag{53}
$$

where:

$$
t_1=\lambda R\left(\frac{\rho}{\alpha_1}\right)^{\frac{1}{2}}\omega\;;\quad s_1=\lambda R\left(\frac{\rho}{\beta_1}\right)^{\frac{1}{2}}\omega\;;\quad s_2=R\left(\frac{\rho}{\beta_1}\right)^{\frac{1}{2}}\omega\,.
$$

Finally we find Q from (43):

$$
Q=\pi\lambda R(\alpha_4(2h)^{-1}V\lambda R-2\alpha_3 C_1 J_1(t_1))\,. \tag{54}
$$

Equations (51)–(54) give the solution when ω and either Q or V are given. At a resonance-frequency $V=0$ and we find ω_r by equating the determinant of the equations (51)–(53) to zero. Hence the determinant of the following matrix must vanish:

$$
\begin{pmatrix}
J_1(t_1) & -J_1(s_1) & -Y_1(s_1)\\
(\alpha_2-\alpha_1)J_1(t_1)+\alpha_1 t_1 J_0(t_1) & -(\beta_2-\beta_1)J_1(s_1)-\beta_1 s_1 J_0(s_1) & -(\beta_2-\beta_1)Y_1(s_1)-\beta_1 s_1 Y_0(s_1)\\
0 & (\beta_2-\beta_1)J_1(s_2)+\beta_1 s_2 J_0(s_2) & (\beta_2-\beta_1)Y_1(s_2)+\beta_1 s_2 Y_0(s_2)
\end{pmatrix}
\tag{55}
$$

At any antiresonance-frequency $Q=0$ and V is eliminated. We find that in the matrix given above only the element in the first column and in the second row must be changed into:

$$
\left(\alpha_2-\alpha_1+2\,\frac{\alpha_3^2}{\alpha_4}\right)J_1(t_1)+\alpha_1 t_1 J_0(t_1)\,. \tag{56}
$$

We find the antiresonance-frequencies ω_a by equating the determinant of this matrix to zero.

6. Experimental Verification

Computations for the partly electroded plate, mentioned at the end of section 5 are performed. Using the notation of section 5 this plate is characterized by:

$$
\left.
\begin{aligned}
&Y=5.8*10^{10}\,\mathrm{Nm}^{-2}\;;\quad \nu=0.39\;;\quad \rho=7.26*10^{3}\,\mathrm{kg\cdot m^{-3}}\;;\\
&d^{3}{}_{.11}=-1.45*10^{-10}\,\mathrm{Coul\cdot N^{-1}}\;;\quad {}^{T}\varepsilon^{33}=1.33*10^{-8}\,\mathrm{Coul\cdot(Vm)^{-1}}\;;\\
&k_1^{\frac{1}{1}}=0.31\;;\quad R=0.025\,\mathrm{m}\,.
\end{aligned}
\right\}
\tag{57}
$$

These values belong to the ceramic P1-60 and were given by the manufactures of the plate [7]. The thickness which does not enter in the theory, was 1 mm.

At each constant value of the ratio of the radius of the electrodes to the radius of the plate λ, the total charge on the upper electrode Q, and hence the capacity of the plate $C=Q/V$, is computed. In figure 4 the behaviour of C as a function of ω is given.

We focus our attention to the two fundamental frequencies: the smallest resonance frequency ω_r at which the capacity C is infinite, and the smallest antiresonance-frequency ω_a, at which the capacity C vanishes. The values of ω_r and ω_a are computed as a function of λ by equating the determinants mentioned at the end of section 5 to zero. These functions are given in figure 5. We compare these functions with measured values of ω_r and ω_a. Therefore we use the circuit given in figure 6. Here P represents the ceramic plate. A and B represent the input channels of a two-beam oscilloscope. The internal capacity of an input channel is 30 pf; they are denoted

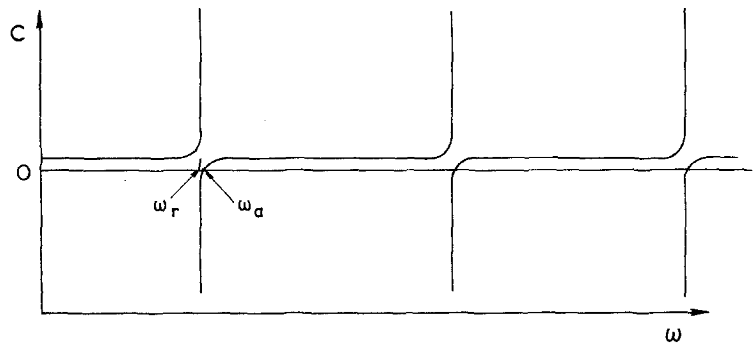

Figure 4. The capacity C of the plate as a function of the frequency ω.

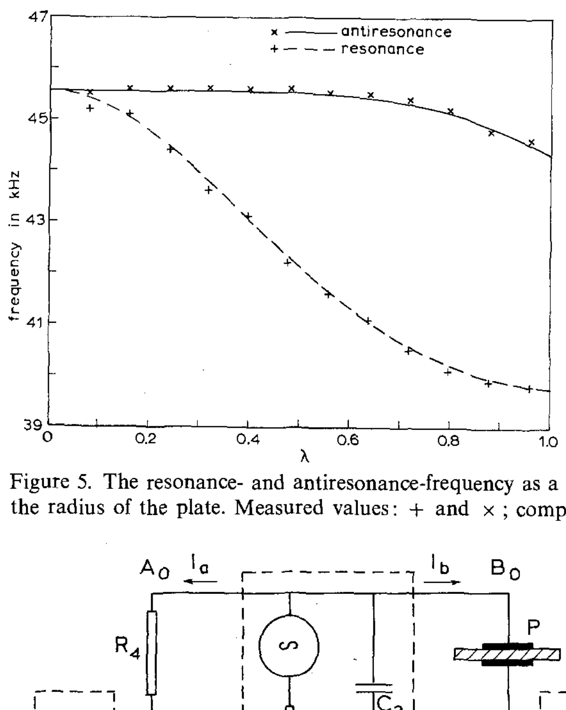

Figure 5. The resonance- and antiresonance-frequency as a function of the ratio of the radius of the electrodes to the radius of the plate. Measured values: + and ×; computed values: —— and - - - -.

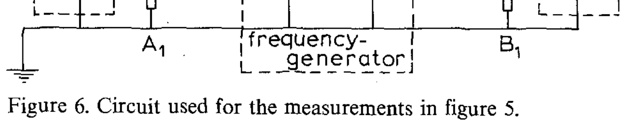

Figure 6. Circuit used for the measurements in figure 5.

by C_1 resp. C_2. The oscilloscope is used in order to measure the current I_A resp. I_B. We used the following resistances: $R_1 = R_2 = 10\ \Omega$ and $R_4 = 1000\Omega$. Finally the frequency-generator possesses an internal resistance $R_3 = 200\ \Omega$ and an internal capacity $C_3 = 25$ pf.

Between A_0 and A_1 the current I_A meets an impedance

$$Z_A = R_4 + \left(\frac{1}{R_1} + i\omega C_1\right)^{-1}, \tag{58}$$

with $i = \sqrt{-1}$. The first resonance-frequency will appear to be about 50 kHz. Hence in this range the phase-delay due to Z_A is less than $2 \cdot 10^{-8}$ rad, and we may consider Z_A to be real.

We consider the electric behaviour of the plate as a frequency-dependent capacity C, parallel-circuited with a resistance R. Hence the impedance between B_0 and B_1 is

$$Z_B = \left(\frac{1}{R} + i\omega C_{(\omega)}\right)^{-1} + \left(\frac{1}{R_2} + i\omega C_2\right)^{-1}. \tag{59}$$

When $\omega < 50$ kHz, the phase-delay due to the second term in the right-hand side of (59) is less than $2 \cdot 10^{-5}$ rad. Hence we may assume:

$$Z_B = \left(\frac{1}{R} + i\omega C_{(\omega)}\right)^{-1} + Z_m \tag{60}$$

with Z_m real.

The impedance Z_B will generally be complex, except for two values of ω, at which Z_B is real:

1) $\omega = \omega_r$, since then $C = \infty$ and $Z_B = Z_m$
2) $\omega = \omega_a$, since then $C = 0$ and $Z_B = Z_m + R$

The values ω_r and ω_a can be read out very accurately as the values of ω at which the two beams of the oscilloscope are in phase. These values are also given in figure 5.

The similarity of the measured and computed values in figure 5 was achieved after a slight modification of two constants:

$$Y = 5.4 * 10^{10}\,\mathrm{Nm}^{-2}\ ; \quad d^3_{.11} = -1.37 * 10^{-10}\,\mathrm{Coul}\cdot\mathrm{N}^{-1}, \tag{61}$$

and hence $k_1^{\frac{1}{1}} = 0.28$. This is a change of less than 10% in the electromechanical coupling factor which may be due to the tooling of the plate.

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